Theoretische Physik IV: Statistische Mechanik und Thermodynamik

Problem Set No. 11

Due on: Friday, 11.7.08 in the practice groups

Exercise 11.1 *(Bose-Einstein Condensation)* **(10 points)**

Consider an ideal Bose gas with total particle number N. The single particle energy levels are normalized such that $E_0 = 0, E_0 \leq E_1 \leq E_2 \leq \ldots$.

(a) Starting from the mean occupation number

$$\langle n_\nu \rangle = \frac{1}{e^{\beta(E_\nu - \mu)} - 1},$$

justify that $\mu \leq 0$. (3 points)

(b) Now assume $\mu < 0$. What do we obtain for the mean occupation number $\langle n_\nu \rangle$ (for arbitrary ν) in the limit $T \to 0$? What kind of problem results for N? (3 points)

(c) Now assume that for $T \to 0$ we always have $0 < -\beta\mu \ll 1$. What do we obtain in this case for the mean occupation number $\langle n_0 \rangle$ of the lowest-lying energy level? How is the problem of (b) solved therewith? (4 points)

Exercise 11.2 *(Photon Gas)* **(10 points)**

In problem 6.2 we already dealt with the photon gas. Here we want to derive its equation of state. Photons are bosons with $z = e^{\beta\mu} = 1$, therefore the grandcanonical partition sum reads

$$\ln Z_{grk} = \frac{pV}{kT} = -\sum_\epsilon \ln\left(1 - e^{-\epsilon/kT}\right)$$

The photons are located in a cubic box of edge length L ($V = L^3$). The energy levels for the single photons are given by the relativistic relation $\epsilon = c|\mathbf{p}|$.

(a) In the case of large V one can replace the sum over the states by an integral. Show that in this case one has to do the following replacement (keep in mind that a photon has two spin states):

$$\sum_\epsilon \to \int \frac{8\pi V}{h^3 c^3} \epsilon^2 d\epsilon$$

(5 points)

(b) Perform the integration and from this calculate the internal energy $\langle E \rangle$. Show that $pV = \frac{1}{3}\langle E \rangle$.
Hint: $\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}$ (5 points)

Exercise 11.3 *(Identical Particles)* **(10 points)**

Consider a system consisting of two identical non-interacting particles. Each particle can take on exactly three distinct energies $\epsilon_1 = -\epsilon, \epsilon_2 = 0, \epsilon_3 = +\epsilon$. The system is in thermal contact to a heat bath of temperature T. Calculate the canonical partition sum of the system and its average energy $\langle E \rangle$ as a function of T (or β) for the cases that both particles are

(a) distinguishable

(b) bosons

(c) fermions

Sketch for the above three cases the corresponding $\langle E \rangle - T$-diagrams.

Bonus Exercise 11.4 *(Modified Fermi Gas)* **(10 extra points)**

Note: This exercise is optional. You may want to solve it to improve your score.

Consider an ideal Fermi Gas whose energy spectrum is given by $\epsilon \sim |\mathbf{p}|^s$ and which is contained in a box of volume V in a space of n dimensions.

(a) Show that for this system $pV = \frac{s}{n} \langle E \rangle$. (6 points)

(b) Express the average particle number $\langle N \rangle$ as a function of β, n, s and appropriate Fermi-Dirac functions $g_\nu(z)$. (2 points)

(c) Express $\langle E \rangle$ as a function of $\langle N \rangle$ and T. (2 points)

Hints:

- The volume of a sphere in n dimensions with radius r is given by $V = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} r^n$.
- The functions $g_\nu(z)$ are the so-called Fermi-Dirac functions $g_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty dx \frac{x^{\nu-1}}{z^{-1}e^x+1}$